Introduction to Calculus

TERMINOLOGY

**Composite function:** A function of a function. One function, \( f(x) \), is a composite of one function to another function, for example \( g(x) \)

**Continuity:** Describing a line or curve that is unbroken over its domain

**Continuous function:** A function is continuous over an interval if it has no break in its graph. For every \( x \) value on the graph the limit exists and equals the function value

**Derivative at a point:** This is the gradient of a curve at a particular point

**Derivative function:** The gradient function of a curve obtained through differentiation

**Differentiable function:** A function which is continuous and where the gradient exists at all points on the function

**Differentiation:** The process of finding the gradient of a tangent to a curve which is called the derivative

**Differentiation from first principles:** The process of finding the gradient of a tangent to a curve by finding the gradient of the secant between two points and finding the limit as the secant becomes a tangent

**Gradient of a secant:** The gradient (slope) of the line between two points that lies close together on a function

**Gradient of a tangent:** The gradient (slope) of a line that is a tangent to the curve at a point on a function. It is the derivative of the function

**Rate of change:** The rate at which the dependent variable changes as the independent variable changes
INTRODUCTION

CALCULUS IS A VERY IMPORTANT part of mathematics and involves the measurement of change. It can be applied to many areas such as science, economics, engineering, astronomy, sociology and medicine. We also see articles in newspapers every day that involve change: the spread of infectious diseases, population growth, inflation, unemployment, filling of our water reservoirs.

For example, this graph shows the change in crude oil production in Iran over the years. Notice that while the graph shows that production is increasing over recent years, the rate at which it is being produced seems to be slowing down. Calculus is used to look at these trends and predict what will happen in the future.

There are two main branches of calculus. Differentiation is used to calculate the rate at which two variables change in relation to one another. Anti-differentiation, or integration, is the inverse of differentiation and uses information about rates of change to go back and examine the original variables. Integration can also be used to find areas of curved objects.

DID YOU KNOW?

‘Calculus’ comes from the Latin meaning pebble or small stone. In ancient civilisations, stones were used for counting. However, the mathematics practised by these early people was quite sophisticated. For example, the ancient Greeks used sums of rectangles to estimate areas of curved figures.

However, it wasn’t until the 17th century that there was a breakthrough in calculus when scientists were searching for ways of measuring motion of objects such as planets, pendulums and projectiles.

Isaac Newton, an Englishman, discovered the main principles of calculus when he was 23 years old. At this time an epidemic of bubonic plague closed Cambridge University where he was studying, so many of his discoveries were made at home.

He first wrote about his calculus methods, which he called fluxions, in 1671, but his Method of fluxions was not published until 1704.

Gottfried Leibniz (1646–1716), in Germany, was also studying the same methods and there was intense rivalry between the two countries over who was first!

Search the Internet for further details on these two famous mathematicians. You can find out about the history of calculus and why it was necessary for mathematicians all those years ago to invent it.
In this chapter you will learn about differentiation, which measures the rate of change of one variable with respect to another.

**Gradient**

**Gradient of a straight line**

The gradient of a straight line measures its slope. You studied gradient in the last chapter.

\[ m = \frac{\text{rise}}{\text{run}} \]

**Class Discussion**

Remember that an **increasing** line has a positive gradient and a **decreasing** line has a negative gradient.

\[ \text{positive} \quad \text{negative} \]

Notice also that a horizontal line has zero gradient. Can you see why?

Can you find the gradient of a vertical line? Why?

Gradient plays an important part, not just in mathematics, but in many areas including science, business, medicine and engineering. It is used everywhere we want to find rates.

On a graph, the gradient measures the rate of change of the dependent variable with respect to the change in the independent variable.
EXAMPLES

1. The graph shows the average distance travelled by a car over time. Find the gradient and describe it as a rate.

 SOLUTION

The line is increasing so it will have a positive gradient.

\[
m = \frac{\text{rise}}{\text{run}} = \frac{400}{5} = 80 = 80
\]

This means that the car is travelling at the rate of 80 km/hour.

2. The graph shows the number of cases of flu reported in a town over several weeks.

Find the gradient and describe it as a rate.
Solution

The line is decreasing so it will have a negative gradient.

\[ m = \frac{\text{rise}}{\text{run}} = \frac{-1500}{10} = -\frac{150}{1} = -150 \]

This means that the rate is \(-150\) cases/week, or the number of cases reported is decreasing by 150 cases/week.

When finding the gradient of a straight line in the number plane, we think of a change in \(y\) values as \(x\) changes. The gradients in the examples above show rates of change.

However, in most examples in real life, the rate of change will vary. For example, a car would speed up and slow down depending on where it is in relation to other cars, traffic light signals and changing speed limits.

Gradient of a curve

Class Discussion

The two graphs show the distance that a bicycle travels over time. One is a straight line and the other is a curve.

Is the average speed of the bicycle the same in both cases? What is different about the speed in the two graphs?

How could you measure the speed in the second graph at any one time? Does it change? If so, how does it change?
Here is a more general curve. What could you say about its gradient? How does it change along the curve?

Copy the graph and mark on it where the gradient is positive, negative and zero.

Using what we know about the gradient of a straight line, we can see where the gradient of a curve is positive, negative or zero by drawing tangents to the curve in different places around the curve.

Notice that when the curve increases it has a positive gradient, when it decreases it has a negative gradient and when it turns around the gradient is zero.

**Investigation**

There are some excellent computer programs that will draw tangents to a curve and then sketch the gradient curve. One of these is Geometer Sketchpad.

Explore how to sketch gradient functions using this or a similar program as you look at the examples below.
EXAMPLES

Describe the gradient of each curve.

1. Where the curve increases, the gradient is positive. Where it decreases, it is negative. Where it turns around, it has a zero gradient.

Solution

2. Where the curve increases, the gradient is positive. Where it decreases, it is negative. Where it turns around, it has a zero gradient.

Solution
Since we have a formula for finding the gradient of a straight line, we find the gradient of a curve by measuring the gradient of a tangent to the curve.

**EXAMPLE**

(a) Make an accurate sketch of \( y = x^2 \) on graph paper.
(b) Draw tangents to this curve at the points where \( x = -3, x = -2, x = -1, x = 0, x = 1, x = 2 \) and \( x = 3 \).
(c) Find the gradient of each of these tangents.
(d) Draw the graph of the gradients (the gradient function) on a number plane.

**Solution**

(a) and (b)

![Graph of y = x^2 with tangents](image)

(c) At \( x = -3, m = -6 \)
    At \( x = -2, m = -4 \)
    At \( x = -1, m = -2 \)
    At \( x = 0, m = 0 \)
    At \( x = 1, m = 2 \)
    At \( x = 2, m = 4 \)
    At \( x = 3, m = 6 \)

(d) ![Graph of gradients](image)
Drawing tangents to a curve is difficult. We can do a rough sketch of the gradient function of a curve without knowing the actual values of the gradients of the tangents.

To do this, notice in the example above that where \( m \) is positive, the gradient function is above the \( x \)-axis, where \( m = 0 \), the gradient function is on the \( x \)-axis and where \( m \) is negative, the gradient function is below the \( x \)-axis.

**EXAMPLES**

Sketch the gradient function of each curve.

1. 

![Gradient Function Graph](image)

**Solution**

First we mark in where the gradient is positive, negative and zero.

![Gradient Function Graph](image)

Now on the gradient graph, place the points where \( m = 0 \) on the \( x \)-axis. These are at \( x_1 \), \( x_2 \) and \( x_3 \).
To the left of $x_1$, the gradient is negative, so this part of the graph will be below the $x$-axis. Between $x_1$ and $x_2$, the gradient is positive, so the graph will be above the $x$-axis. Between $x_2$ and $x_3$, the gradient is negative, so the graph will be below the $x$-axis. To the right of $x_3$, the gradient is positive, so this part of the graph will be above the $x$-axis.

2. **Solution**

First mark in where the gradient is positive, negative and zero.
The gradient is zero at \( x_1 \) and \( x_2 \). These points will be on the \( x \)-axis. To the left of \( x_1 \), the gradient is positive, so this part of the graph will be above the \( x \)-axis. Between \( x_1 \) and \( x_2 \), the gradient is negative, so the graph will be below the \( x \)-axis. To the right of \( x_2 \), the gradient is positive, so this part of the graph will be above the \( x \)-axis.

8.1 Exercises

Sketch the gradient function for each graph.

1. 
2. 
3. 
4. 
5. 
6.
Differentiation from First Principles

Seeing where the gradient of a curve is positive, negative or zero is a good first step, but there are methods to find a formula for the gradient of a tangent to a curve.

The process of finding the gradient of a tangent is called **differentiation**. The resulting function is called the **derivative**.

**Differentiability**

A function is called a **differentiable function** if the gradient of the tangent can be found.

There are some graphs that are not differentiable in places.

Most functions are **continuous**, which means that they have a smooth unbroken line or curve. However, some have a gap, or discontinuity, in the graph (e.g. hyperbola). This can be shown by an asymptote or a ‘hole’ in the graph. We cannot find the gradient of a tangent to the curve at a point that doesn’t exist! So the function is not differentiable at the point of discontinuity.
A function may be continuous but not smooth. It may have a sharp corner. Can you see why curves are not differentiable at the point where there is a corner?

\[ y = f(x) \]

This function is not differentiable at \( b \) as the curve is discontinuous at this point.

A function is differentiable at the point \( x = a \) if the derivative exists at that point. This can only happen if the function is continuous and smooth at \( x = a \).
1. Find all points where the function below is not differentiable.

![Graph showing a function with points A, B, and C marked]

**Solution**

The function is not differentiable at points A and B since there are sharp corners and the curve is not smooth at these points.

It is not differentiable at point C since the function is discontinuous at this point.

2. Is the function $f(x) = \begin{cases} x^2 & \text{for } x \geq 1 \\ 3x - 2 & \text{for } x < 1 \end{cases}$ differentiable at all points?

**Solution**

The functions $f(x) = x^2$ and $f(x) = 3x - 2$ are both differentiable at all points.

However, we need to look at where one finishes and the other starts, at $f(1)$.

For $f(x) = x^2$

\[
f(1) = 1^2 = 1
\]

For $f(x) = 3x - 2$

\[
f(1) = 3(1) - 2 = 1
\]

This means that both pieces of this function join up (the function is continuous). However, to be differentiable, the curve must be smooth at this point.
Sketching this function shows that it is not smooth (it has a sharp corner) so it is not differentiable at $x = 1$.

8.2 Exercises

For each function, state whether it has any points at which it is not differentiable.

1.  

3.  

2.  

4.
5.

6. \( f(x) = \frac{4}{x} \)

7. \( y = -\frac{1}{x + 3} \)

8. \( f(x) = \begin{cases} x^3 & \text{if } x > 2 \\ x + 1 & \text{if } x \leq 2 \\ 2x & \text{for } x > 3 \end{cases} \)

9. \( f(x) = \begin{cases} 3 & \text{for } -2 \leq x \leq 3 \\ 1 - x^2 & \text{for } x < -2 \end{cases} \)

10.

11. \( y = \tan x \) for \( 0^\circ \leq x \leq 360^\circ \)

12. \( f(x) = \frac{|x|}{x} \)

13. \( f(\theta) = -3 \cos 2\theta \)

14. \( g(\phi) = \sin^2 \phi \)

15. \( y = \frac{x - 3}{x^2 - 9} \)

**Limits**

To differentiate from first principles, we need to look more closely at the concept of a limit.

A limit is used when we want to move as close as we can to something. Often this is to find out where a function is near a gap or discontinuous point. You saw this in Chapter 5 when looking at discontinuous graphs. In this topic, it is used when we want to move from a gradient of a line between two points to a gradient of a tangent.

**EXAMPLES**

1. Find \( \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} \).

**Solution**

\[
\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x + 1)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 1) = 2 + 1 = 3
\]
2. Find an expression in terms of $x$ for $\lim_{h \to 0} \frac{2xh - h^2 - 3h}{h}$.

**Solution**

\[
\lim_{h \to 0} \frac{2xh - h^2 - 3h}{h} = \lim_{h \to 0} \frac{h(2x - h - 3)}{h} = \lim_{h \to 0} (2x - h - 3) = 2x - 3
\]

3. Find an expression in terms of $x$ for $\lim_{\delta x \to 0} \frac{3x^2 \delta x + \delta x^2 - 5\delta x}{\delta x}$.

**Solution**

\[
\lim_{\delta x \to 0} \frac{3x^2 \delta x + \delta x^2 - 5\delta x}{\delta x} = \lim_{\delta x \to 0} \frac{\delta x(3x^2 + \delta x - 5)}{\delta x} = \lim_{\delta x \to 0} (3x^2 + \delta x - 5) = 3x^2 - 5
\]

8.3 Exercises

1. Evaluate

(a) $\lim_{x \to 0} \frac{x^2 + 3x}{x}$
(b) $\lim_{x \to 0} \frac{5x^3 - 2x^2 - 7x}{x}$
(c) $\lim_{x \to 3} \frac{x^2 - 3x}{x - 3}$
(d) $\lim_{t \to 4} \frac{t^2 - 16}{t - 4}$
(e) $\lim_{g \to 1} \frac{g^2 - 1}{g - 1}$
(f) $\lim_{x \to 2} \frac{x^2 + x - 2}{x + 2}$
(g) $\lim_{h \to 0} \frac{h^5 + 2h}{h}$
(h) $\lim_{x \to 3} \frac{x^2 - 7x + 12}{x - 3}$
(i) $\lim_{n \to 5} \frac{n^2 - 25}{n - 5}$
(j) $\lim_{x \to 1} \frac{x^2 + 4x + 3}{x^2 - 1}$

2. Find an expression in terms of $x$

(a) $\lim_{h \to 0} \frac{x^2 h - 2xh - 4h}{h}$
(b) $\lim_{h \to 0} \frac{2x^3 h + xh - h}{h}$
(c) $\lim_{h \to 0} \frac{3x^2 h^2 - 7xh + 4h^2 - h}{h}$
(d) $\lim_{h \to 0} \frac{4x^4 h - x^2 h - 4xh^2}{h}$
(e) $\lim_{h \to 0} \frac{x^2 h^2 + 3xh^2 - 4xh + 3h}{h}$
(f) $\lim_{h \to 0} \frac{2x^2 h + 5xh^2 + 6h}{h}$
(g) $\lim_{\delta x \to 0} \frac{x^2 \delta x^2 - 2x\delta x}{\delta x}$
(h) $\lim_{\delta x \to 0} \frac{4x^2 \delta x^2 - 2\delta x}{\delta x}$
(i) $\lim_{\delta x \to 0} \frac{x^3 \delta x^2 + 3x\delta x - \delta x}{\delta x}$
(j) $\lim_{\delta x \to 0} \frac{x^2 \delta x - 2x\delta x + 9\delta x}{\delta x}$
Differentiation as a limit

The formula \( m = \frac{y_2 - y_1}{x_2 - x_1} \) is used to find the gradient of a straight line when we know two points on the line. However, when the line is a tangent to a curve, we only know one point on the line—the point of contact with the curve.

To differentiate from first principles, we first use the point of contact and another point close to it on the curve (this line is called a secant) and then we move the second point closer and closer to the point of contact until they overlap and the line is at single point (the tangent). To do this, we use a limit.

If you look at a close up of a graph, you can get some idea of this concept. When the curve is magnified, two points appear to be joined by a straight line. We say the curve is locally straight.

Investigation

Use a graphics calculator or a computer program to sketch a curve and then zoom in on a section of the curve to see that it is locally straight.

For example, here is a parabola.

Notice how it looks straight when we zoom in on a point on the parabola?

Use technology to sketch other curves and zoom in to show that they are locally straight.
Before using limits to find different formulae for differentiating from first principles, here are some examples of how we can calculate an approximate value for the gradient of the tangent to a curve. By taking two points close together, as in the example below, we find the gradient of the secant and then estimate the gradient of the tangent.

**EXAMPLES**

1. For the function \( f(x) = x^3 \), find the gradient of the secant \( PQ \) where \( P \) is the point on the function where \( x = 2 \) and \( Q \) is another point on the curve close to \( P \). Choose different values for \( Q \) and use these results to estimate the gradient of the curve at \( P \).
Solution

\( P = (2, f(2)) \)

Take different values of \( x \) for point \( Q \), for example \( x = 2.1 \)

Using different values of \( x \) for point \( Q \) gives the results in the table.

<table>
<thead>
<tr>
<th>Point ( Q )</th>
<th>Gradient of secant ( PQ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2.1, f(2.1)))</td>
<td>[ m = \frac{f(2.1) - f(2)}{2.1 - 2} = \frac{2.1^3 - 2^3}{2.1 - 2} = 12.61 ]</td>
</tr>
<tr>
<td>((2.01, f(2.01)))</td>
<td>[ m = \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{2.01^3 - 2^3}{2.01 - 2} = 12.0601 ]</td>
</tr>
<tr>
<td>((2.001, f(2.001)))</td>
<td>[ m = \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{2.001^3 - 2^3}{2.001 - 2} = 12.006001 ]</td>
</tr>
<tr>
<td>((1.9, f(1.9)))</td>
<td>[ m = \frac{f(1.9) - f(2)}{1.9 - 2} = \frac{1.9^3 - 2^3}{1.9 - 2} = 11.41 ]</td>
</tr>
<tr>
<td>((1.99, f(1.99)))</td>
<td>[ m = \frac{f(1.99) - f(2)}{1.99 - 2} = \frac{1.99^3 - 2^3}{1.99 - 2} = 11.9401 ]</td>
</tr>
<tr>
<td>((1.999, f(1.999)))</td>
<td>[ m = \frac{f(1.999) - f(2)}{1.999 - 2} = \frac{1.999^3 - 2^3}{1.999 - 2} = 11.994001 ]</td>
</tr>
</tbody>
</table>

From these results, a good estimate for the gradient at \( P \) is 12.

We can say that as \( x \) approaches 2, the gradient approaches 12.

We can write \( \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = 12 \).
2. For the curve $y = x^2$, find the gradient of the secant $AB$ where $A$ is the point on the curve where $x = 5$ and point $B$ is close to $A$. Find an estimate of the gradient of the curve at $A$ by using three different values for $B$.

**Solution**

$A = (5, f(5))$

Take three different values of $x$ for point $B$, for example $x = 4.9, x = 5.1$ and $x = 5.01$.

(a) $B = (4.9, f(4.9))$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(4.9) - f(5)}{4.9 - 5} = \frac{4.9^2 - 5^2}{4.9 - 5} = 9.9$$

(b) $B = (5.1, f(5.1))$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(5.1) - f(5)}{5.1 - 5} = \frac{5.1^2 - 5^2}{5.1 - 5} = 10.1$$

(c) $B = (5.01, f(5.01))$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(5.01) - f(5)}{5.01 - 5} = \frac{5.01^2 - 5^2}{5.01 - 5} = 10.01$$

From these results, a good estimate for the gradient at $A$ is 10.

We can say that as $x$ approaches 5, the gradient approaches 10.

We can write $\lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = 10$. 

We can find a general formula for differentiating from first principles by using $c$ rather than any particular number. We use general points $P(c, f(c))$ and $Q(x, f(x))$ where $x$ is close to $c$.

The gradient of the secant $PQ$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x) - f(c)}{x - c}$$
The gradient of the tangent at \( P \) is found when \( x \) approaches \( c \). We call this \( f'(c) \).

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

There are other versions of this formula.

We can call the points \( P(x, f(x)) \) and \( Q(x + h, f(x + h)) \) where \( h \) is small.

Secant \( PQ \) has gradient

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x + h) - f(x)}{x + h - x} = \frac{f(x + h) - f(x)}{h}
\]

To find the gradient of the secant, we make \( h \) smaller as shown, so that \( Q \) becomes closer and closer to \( P \).
As \( h \) approaches 0, the gradient of the tangent becomes \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).
We call this \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

If we use \( P(x, y) \) and \( Q(x + \delta x, y + \delta y) \) close to \( P \) where \( \delta x \) and \( \delta y \) are small:

Gradient of secant \( PQ \)
\[
m = \frac{y_2 - y_1}{x_2 - x_1}
= \frac{y + \delta y - y}{x + \delta x - x}
= \frac{\delta y}{\delta x}
\]

As \( \delta x \) approaches 0, the gradient of the tangent becomes \( \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \). We call this \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}
\]

All of these different notations stand for the derivative, or the gradient of the tangent:
\[
\frac{dy}{dx}, \frac{d}{dx}(y), \frac{d}{dx}(f(x)), f'(x), y'
\]

These occur because Newton, Leibniz and other mathematicians over the years have used different notation.

**Investigation**

Leibniz used \( \frac{dy}{dx} \) where \( d \) stood for ‘difference’. Can you see why he would have used this?

Use the Internet to explore the different notations used in calculus and where they came from.
The three formulae for differentiating from first principles all work in a similar way.

**EXAMPLE**

Differentiate from first principles to find the gradient of the tangent to the curve $y = x^2 + 3$ at the point where $x = 1$.

**Solution**

**Method 1:**

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$f(x) = x^2 + 3$

$f(1) = 1^2 + 3 = 4$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x^2 + 3) - 4}{x - 1}$$

$$= \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1)$$

$$= 1 + 1$$

$$= 2$$

**Method 2:**

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$f(x) = x^2 + 3$

$f(1) = 1^2 + 3 = 4$

$f(x + h) = (x + h)^2 + 3$

When $x = 1$

$f(1 + h) = (1 + h)^2 + 3$

$$= 1 + 2h + h^2 + 3$$

$$= 2h + h^2 + 4$$
\[
\frac{f'(x)}{f'(1)} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} h(2 + h) = \lim_{h \to 0} (2 + h) = 2 + 0 = 2
\]

Method 3:

\[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}
\]

\[y = x^2 + 3\]

When \(x = 1\)
\[y = 1^2 + 3 = 4\]

So point \((1, 4)\) lies on the curve.

Substitute point \((1 + \delta x, 4 + \delta y)\):

\[4 + \delta y = (1 + \delta x)^2 + 3 = 1 + 2\delta x + \delta x^2 + 3 = 2\delta x + \delta x^2 + 4\]
\[\delta y = 2\delta x + \delta x^2\]
\[\frac{\delta y}{\delta x} = \frac{2\delta x + \delta x^2}{\delta x} = \frac{\delta x(2 + \delta x)}{\delta x} = 2 + \delta x\]
\[\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (2 + \delta x) = 2 + 0 = 2\]

We can also use these formulae to find the derivative function generally.
### Example

Differentiate \( f(x) = 2x^2 + 7x - 3 \) from first principles.

**Solution**

\[
\begin{align*}
  f(x) &= 2x^2 + 7x - 3 \\
  f(x + h) &= 2(x + h)^2 + 7(x + h) - 3 \\
  &= 2(x^2 + 2xh + h^2) + 7x + 7h - 3 \\
  &= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 \\
  f(x + h) - f(x) &= (2x^2 + 4xh + 2h^2 + 7x + 7h - 3) - (2x^2 + 7x - 3) \\
  &= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3 \\
  &= 4xh + 2h^2 + 7h \\
  f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
  &= \lim_{h \to 0} \frac{4xh + 2h^2 + 7h}{h} \\
  &= \lim_{h \to 0} \frac{h(4x + 2h + 7)}{h} \\
  &= \lim_{h \to 0} (4x + 2h + 7) \\
  &= 4x + 0 + 7 \\
  &= 4x + 7
\end{align*}
\]

### Exercises

1. (a) Find the gradient of the secant between the point \((1, 2)\) and the point where \(x = 1.01\), on the curve \(y = x^4 + 1\).
(b) Find the gradient of the secant between \((1, 2)\) and the point where \(x = 0.999\) on the curve.
(c) Use these results to find the gradient of the tangent to the curve \(y = x^4 + 1\) at the point \((1, 2)\).

2. A function \(f(x) = x^3 + x\) has a tangent at the point \((2, 10)\).
   (a) Find the value of \(\frac{f(x) - f(2)}{x - 2}\) when \(x = 2.1\).
   (b) Find the value of \(\frac{f(x) - f(2)}{x - 2}\) when \(x = 2.01\).
   (c) Evaluate \(\frac{f(x) - f(2)}{x - 2}\) when \(x = 1.99\).
   (d) Hence find the gradient of the tangent at the point \((2, 10)\).

3. For the function \(f(x) = x^2 - 4\), find the derivative at point \(P\) where \(x = 3\) by selecting points near \(P\) and finding the gradient of the secant.

4. If \(f(x) = x^2\),
   (a) find \(f(x + h)\)
   (b) show that \(f(x + h) - f(x) = 2xh + h^2\)
5. A function is given by $f(x) = 2x^2 - 7x + 3$.
   (a) Show that $f(x + h) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3$.
   (b) Show that $f(x + h) - f(x) = 4xh + 2h^2 - 7h$.
   (c) Show that $f(x + h) - f(x) \over h = 4x + 2h - 7$.
   (d) Find $f'(x)$.

6. A function is given by $f(x) = x^2 + x + 5$.
   (a) Find $f(2)$.
   (b) Find $f(2 + h)$.
   (c) Find $f(2 + h) - f(2)$.
   (d) Show that $f(2 + h) - f(2) \over h = 5 + h$.
   (e) Find $f'(2)$.

7. Given the curve $f(x) = 4x^3 - 3$.
   (a) find $f(-1)$
   (b) find $f(-1 + h) - f(-1)$
   (c) find the gradient of the tangent to the curve at the point where $x = -1$.

8. For the parabola $y = x^2 - 1$
   (a) find $f(3)$
   (b) find $f(3 + h) - f(3)$
   (c) find $f'(3)$.

9. For the function $f(x) = 4 - 3x - 5x^2$
   (a) find $f'(1)$
   (b) similarly, find the gradient of the tangent at the point $(-2, -10)$.

10. For the parabola $y = x^2 + 2x$
    (a) show that $\delta y = 2x\delta x + \delta x^2 + 2\delta x$ by substituting the point $(x + \delta x, y + \delta y)$
    (b) show that $\delta y \over \delta x = 2x + \delta x + 2$
    (c) find $dy \over dx$.

11. Differentiate from first principles to find the gradient of the tangent to the curve
    (a) $f(x) = x^2$ at the point where $x = 1$
    (b) $y = x^2 + x$ at the point $(2, 6)$
    (c) $f(x) = 2x^2 - 5$ at the point where $x = -3$
    (d) $y = 3x^2 + 3x + 1$ at the point where $x = 2$
    (e) $f(x) = x^3 - 7x - 4$ at the point $(-1, 6)$.

12. Find the derivative function for each curve by differentiating from first principles
    (a) $f(x) = x^2$
    (b) $y = x^2 + 5x$
    (c) $f(x) = 4x^2 - 4x - 3$
    (d) $y = 5x^2 - x - 1$
    (e) $y = x^3$
    (f) $f(x) = 2x^3 + 5x$
    (g) $y = x^3 - 2x^2 + 3x - 1$
    (h) $f(x) = -2x^3$.

13. The curve $y = \sqrt{x}$ has a tangent drawn at the point $(4, 2)$.
    (a) Evaluate $f(x) - f(4) \over x - 4$ when $x = 3.9$.
    (b) Evaluate $f(x) - f(4) \over x - 4$ when $x = 3.999$.
    (c) Evaluate $f(x) - f(4) \over x - 4$ when $x = 4.01$.

14. For the function $f(x) = x^{-1}$, evaluate $f(x) - f(5) \over x - 5$ when $x = 4.99$. 

Remember that $x^{-1} = {1 \over x}$
Chapter 8
Introduction to Calculus

15. Find the gradient of the tangent to the curve \( y = \frac{4}{x^2} \) at point \( P(2, 1) \) by finding the gradient of the secant between \( P \) and a point close to \( P \).

(b) evaluate \( \frac{f(x) - f(5)}{x - 5} \) when \( x = 5.01 \).
(c) Use these results to find the derivative of the function at the point where \( x = 5 \).

Short Methods of Differentiation

The basic rule

Remember that the gradient of a straight line \( y = mx + b \) is \( m \). The tangent to the line is the line itself, so the gradient of the tangent is \( m \) everywhere along the line.

\[ y = mx + b \]

So if \( y = mx \), \( \frac{dy}{dx} = m \)

\[ \frac{d}{dx}(kx) = k \]

For a horizontal line in the form \( y = k \), the gradient is zero.

So if \( y = k \), \( \frac{dy}{dx} = 0 \)

\[ \frac{d}{dx}(k) = 0 \]
Investigation

Differentiate from first principles:

\[ y = x^2 \]
\[ y = x^3 \]
\[ y = x^4 \]

Can you find a pattern? Could you predict what the result would be for \( x^n \)?

Alternatively, you could find an approximation to the derivative of a function at any point by drawing the graph of \[ y = \frac{f(x + 0.01) - f(x)}{0.01} \].

Use a graphics calculator or graphing computer software to sketch the derivative for these functions and find the equation of the derivative.

Mathematicians working with differentiation from first principles discovered this pattern that enabled them to shorten differentiation considerably!

For example:

When \( y = x^2 \), \( y' = 2x \)

When \( y = x^3 \), \( y' = 3x^2 \)

When \( y = x^4 \), \( y' = 4x^3 \)

\[ \frac{d}{dx}(x^n) = nx^{n-1} \]

Proof

You do not need to know this proof.

\[ f(x) = x^n \]
\[ f(x + h) = (x + h)^n \]
\[ f(x + h) - f(x) = (x + h)^n - x^n \]
\[ = (x + h - x)[(x + h)^{n-1} + (x + h)^{n-2}x + (x + h)^{n-3}x^2 + \ldots + (x + h)\cdot x^{n-2} + x^{n-1}] \]
\[ = h[(x + h)^{n-1} + (x + h)^{n-2}x + (x + h)^{n-3}x^2 + \ldots + (x + h)\cdot x^{n-2} + x^{n-1}] \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{h[(x + h)^{n-1} + (x + h)^{n-2}x + (x + h)^{n-3}x^2 + \ldots + (x + h)\cdot x^{n-2} + x^{n-1}]}{h} \]
\[ = \lim_{h \to 0} [(x + h)^{n-1} + (x + h)^{n-2}x + (x + h)^{n-3}x^2 + \ldots + (x + h)\cdot x^{n-2} + x^{n-1}] \]
\[ = (x)^{n-1} + (x)^{n-2}x + (x)^{n-3}x^2 + \ldots + (x)\cdot x^{n-2} + x^{n-1} \]
\[ = nx^{n-1} \]
EXAMPLE

Differentiate $f(x) = x^7$.

Solution

$f'(x) = 7x^6$

There are some more rules that give us short ways to differentiate functions.

The first one says that if there is a constant in front of the $x$ (we call this a coefficient), then it is just multiplied with the derivative.

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

A more general way of writing this rule is:

$$\frac{d}{dx}(kf(x)) = kf'(x)$$

Proof

$$\frac{d}{dx}(kf(x)) = \lim_{h \to 0} \frac{kf(x + h) - kf(x)}{h}$$

$$= \lim_{h \to 0} k \frac{f(x + h) - f(x)}{h}$$

$$= k \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= kf'(x)$$

EXAMPLE

Find the derivative of $3x^8$.

Solution

If $y = 3x^8$

$$\frac{dy}{dx} = 3 \times 8x^7$$

$$= 24x^7$$
Also, if there are several terms in an expression, we differentiate each one separately. We can write this as a rule:

\[
\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)
\]

**Proof**

You do not need to know this proof.

\[
\frac{d}{dx}(f(x) + g(x)) = \lim_{h \to 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h}
\]

\[
= \lim_{h \to 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(x + h) - f(x) + g(x + h) - g(x)}{h}
\]

\[
= \lim_{h \to 0} \left[ \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right]
\]

\[
= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}
\]

\[
= f'(x) + g'(x)
\]

**EXAMPLE**

Differentiate \(x^3 + x^4\).

**Solution**

\[
\frac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3
\]

Many functions use a combination of these rules.

**EXAMPLES**

Differentiate

1. \(7x\)

**Solution**

\[
\frac{d}{dx}(7x) = 7
\]
2. \( f(x) = x^4 - x^3 + 5 \)

Solution

\[
f'(x) = 4x^3 - 3x^2 + 0 = 4x^3 - 3x^2
\]

3. \( y = 4x^7 \)

Solution

\[
\frac{dy}{dx} = 4 \times 7x^6 = 28x^6
\]

4. If \( f(x) = 2x^5 - 7x^3 + 5x - 4 \), evaluate \( f'(-1) \)

Solution

\[
f'(x) = 10x^4 - 21x^2 + 5
\]
\[
f'(-1) = 10(-1)^4 - 21(-1)^2 + 5 = -6
\]

5. Differentiate \( \frac{3x^2 + 5x}{2x} \)

Solution

Divide by 2x before differentiating.

\[
\frac{3x^2 + 5x}{2x} = \frac{3x^2}{2x} + \frac{5x}{2x} = \frac{3}{2}x^2 + \frac{5}{2}
\]
\[
\frac{dy}{dx} = \frac{3}{2} = 1 \frac{1}{2}
\]

6. Differentiate \( S = 2\pi r^2 + 2\pi rh \) with respect to \( r \).

Solution

We are differentiating with respect to \( r \), so \( r \) is the variable and \( \pi \) and \( h \) are constants.

\[
\frac{dS}{dr} = 2\pi(2r) + 2\pi h = 4\pi r + 2\pi h
\]
8.5 Exercises

1. Differentiate
   (a) \( x + 2 \)
   (b) \( 5x - 9 \)
   (c) \( x^2 + 3x + 4 \)
   (d) \( 5x^2 - x - 8 \)
   (e) \( x^3 + 2x^2 - 7x - 3 \)
   (f) \( 2x^3 - 7x^2 + 7x - 1 \)
   (g) \( 3x^4 - 2x^2 + 5x \)
   (h) \( x^6 - 5x^3 - 2x^4 \)
   (i) \( 2x^5 - 4x^3 + x^2 - 2x + 4 \)
   (j) \( 4x^{10} - 7x^3 \)

2. Find the derivative of
   (a) \( x(2x + 1) \)
   (b) \( (2x - 3)^2 \)
   (c) \( (x + 4)(x - 4) \)
   (d) \( (2x^2 - 3)^2 \)
   (e) \( (2x + 5)(x^2 - x + 1) \)

3. Differentiate
   (a) \( \frac{x^2}{6} - x \)
   (b) \( \frac{x^4}{2} - \frac{x^3}{3} + 4 \)
   (c) \( \frac{1}{3}x^6(x^2 - 3) \)
   (d) \( \frac{2x^3 + 5x}{x} \)
   (e) \( \frac{x^2 + 2x}{4x} \)
   (f) \( \frac{2x^5 - 3x^4 + 6x^3 - 2x^2}{3x^2} \)

4. Find \( f'(x) \) when
   \( f(x) = 8x^2 - 7x + 4. \)

5. If \( y = x^4 - 2x^3 + 5, \) find \( \frac{dy}{dx} \) when \( x = -2. \)

6. Find \( \frac{dy}{dx} \) if
   \( y = 6x^{10} - 5x^8 + 7x^3 - 3x + 8. \)

7. If \( s = 5t^2 - 20t, \) find \( \frac{ds}{dt}. \)

8. Find \( g'(x) \) given \( g(x) = 5x^{-4}. \)

9. Find \( \frac{dv}{dt} \) when \( v = 15t^2 - 9. \)

10. If \( h = 40t - 2t^2, \) find \( \frac{dh}{dt}. \)

11. Given \( V = \frac{4}{3} \pi r^3, \) find \( \frac{dV}{dr}. \)

12. If \( f(x) = 2x^3 - 3x + 4, \) evaluate \( f'(1). \)

13. Given \( f(x) = x^2 - x + 5, \) evaluate
    (a) \( f'(3) \)
    (b) \( f'(-2) \)
    (c) \( x \) when \( f'(x) = 7 \)

14. If \( y = x^3 - 7, \) evaluate
    (a) \( \frac{dy}{dx} \) when \( x = 2 \)
    (b) \( x \) when \( \frac{dy}{dx} = 12 \)

15. Evaluate \( g'(2) \) when
    \( g(t) = 3t^3 - 4t^2 - 2t + 1. \)
Tangents and Normals

DID YOU KNOW?

• The word tangent comes from the Latin ‘tangens’, meaning ‘touching’. A tangent to a circle intersects it only once.

• However, a tangent to a curve could intersect the curve more than once.

• A line may only intersect a curve once but not be a tangent.

• So a tangent to a curve is best described as the limiting position of the secant \( PQ \) as \( Q \) approaches \( P \).

Remember from earlier in the chapter that the derivative is the gradient of the tangent to a curve.

\[
\frac{dy}{dx}\text{ is the gradient of the tangent to a curve}
\]
EXAMPLES

1. Find the gradient of the tangent to the parabola \( y = x^2 + 1 \) at the point \((1, 2)\).

**Solution**

\[
\frac{dy}{dx} = 2x + 0
\]

\[
= 2x
\]

At \((1, 2)\) \( \frac{dy}{dx} = 2(1) \)

\[
= 2
\]

So the gradient of the tangent at \((1, 2)\) is 2.

2. Find values of \(x\) for which the gradient of the tangent to the curve \( y = 2x^3 - 6x^2 + 1 \) is equal to 18.

**Solution**

\[
\frac{dy}{dx} = 6x^2 - 12x
\]

\( \frac{dy}{dx} \) is the gradient of the tangent, so substitute \( \frac{dy}{dx} = 18 \).

\[
18 = 6x^2 - 12x
\]

\[
0 = 6x^2 - 12x - 18
\]

\[
= x^2 - 2x - 3
\]

\[
= (x - 3)(x + 1)
\]

\[
x - 3 = 0, x + 1 = 0
\]

\[
\therefore x = 3, \quad x = -1
\]

3. Find the equation of the tangent to the curve \( y = x^4 - 3x^3 + 7x - 2 \) at the point \((2, 4)\).

**Solution**

\[
\frac{dy}{dx} = 4x^3 - 9x^2 + 7
\]

At \((2, 4)\) \( \frac{dy}{dx} = 4(2)^3 - 9(2)^2 + 7 \)

\[= 3\]

So the gradient of the tangent at \((2, 4)\) is 3.

Equation of the tangent:

\[
y - y_1 = m(x - x_1)
\]

\[
y - 4 = 3(x - 2)
\]
The normal is a straight line perpendicular to the tangent at the same point of contact with the curve.

If lines with gradients $m_1$ and $m_2$ are perpendicular, then $m_1m_2 = -1$

**EXAMPLES**

1. Find the gradient of the normal to the curve $y = 2x^2 - 3x + 5$ at the point where $x = 4$.

**Solution**

$\frac{dy}{dx}$ is the gradient of the tangent.

$\frac{dy}{dx} = 4x - 3$

When $x = 4$

$\frac{dy}{dx} = 4 \times 4 - 3$

$= 13$

So $m_1 = 13$

The normal is perpendicular to the tangent.

So $m_1m_2 = -1$
13m_2 = -1

m_2 = -\frac{1}{13}

So the gradient of the normal is \(-\frac{1}{13}\).

2. Find the equation of the normal to the curve \(y = x^3 + 3x^2 - 2x - 1\) at the point \((-1, 3)\).

Solution

\(\frac{dy}{dx}\) is the gradient of the tangent.

\[
\frac{dy}{dx} = 3x^2 + 6x - 2
\]

When \(x = -1\)

\[
\frac{dy}{dx} = 3(-1)^2 + 6(-1) - 2 = -5
\]

So \(m_1 = -5\)

The normal is perpendicular to the tangent.

So \(m_1m_2 = -1\)

\(-5m_2 = -1\)

\(m_2 = \frac{1}{5}\)

So the gradient of the normal is \(\frac{1}{5}\).

Equation of the normal:

\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = \frac{1}{5}(x - (-1))
\]

\[
5y - 15 = x + 1
\]

\[
0 = x - 5y + 16
\]

8.6 Exercises

1. Find the gradient of the tangent to the curve

   (a) \(y = x^3 - 3x\) at the point where \(x = 5\)
   (b) \(f(x) = x^2 + x - 4\) at the point \((-7, 38)\)
   (c) \(f(x) = 5x^4 - 4x - 1\) at the point where \(x = -1\)
   (d) \(y = 5x^2 + 2x + 3\) at the point \((-2, 19)\)
   (e) \(y = 2x^9\) at the point where \(x = 1\)
   (f) \(f(x) = x^3 - 7\) at the point where \(x = 3\)
   (g) \(v = 2t^2 + 3t - 5\) at the point where \(t = 2\)
   (h) \(Q = 3r^3 - 2r^2 + 8r - 4\) at the point where \(r = 4\)
   (i) \(h = t^4 - 4t\) where \(t = 0\)
   (j) \(f(t) = 3t^5 - 8t^3 + 5t\) at the point where \(t = 2\).
2. Find the gradient of the normal to the curve
   (a) \( f(x) = 2x^3 + 2x - 1 \) at the point where \( x = -2 \)
   (b) \( y = 3x^2 + 5x - 2 \) at the point \((-5, 48)\)
   (c) \( f(x) = x^2 - 2x - 7 \) at the point where \( x = -9 \)
   (d) \( y = x^3 + x^2 + 3x - 2 \) at the point \((-4, -62)\)
   (e) \( f(x) = x^{10} \) at the point where \( x = -1 \)
   (f) \( y = x^2 + 7x - 5 \) at the point \((-7, -5)\)
   (g) \( A = 2x^4 + 3x^2 - x + 1 \) at the point where \( x = 3 \)
   (h) \( f(a) = 3a^2 - 2a - 6 \) at the point where \( a = -3 \)
   (i) \( V = h^3 - 4h + 9 \) at the point \((2, 9)\)
   (j) \( g(x) = x^4 - 2x^2 + 5x - 3 \) at the point where \( x = -1 \).

3. Find the gradient of the (i) tangent and (ii) normal to the curve
   (a) \( y = x^2 + 1 \) at the point \((3, 10)\)
   (b) \( f(x) = 5 - x^2 \) at the point where \( x = -4 \)
   (c) \( y = 2x^3 - 7x^2 + 4 \) at the point where \( x = -1 \)
   (d) \( p(x) = x^6 - 3x^2 - 2x + 8 \)
   where \( x = 1 \)
   (e) \( f(x) = 4 - x - x^2 \) at the point \((-6, 26)\).

4. Find the equation of the tangent to the curve
   (a) \( y = x^4 - 5x + 1 \) at the point \((2, 7)\)
   (b) \( f(x) = 5x^3 - 3x^2 - 2x + 6 \) at the point \((1, 6)\)
   (c) \( y = x^2 + 2x - 8 \) at the point \((-3, -5)\)
   (d) \( y = 3x^4 + 1 \) at the point where \( x = 2 \)
   (e) \( v = 4t^4 - 7t^3 - 2 \) at the point where \( t = 2 \)

5. Find the equation of the normal to the curve
   (a) \( f(x) = x^3 - 3x + 5 \) at the point \((3, 23)\)
   (b) \( y = x^2 - 4x - 5 \) at the point \((-2, 7)\)
   (c) \( f(x) = 7x - 2x^2 \) at the point where \( x = 6 \)
   (d) \( y = 7x^2 - 3x - 2 \) at the point \((-3, 70)\)
   (e) \( y = x^4 - 2x^3 + 4x + 1 \) at the point where \( x = 1 \).

6. Find the equation of the (i) tangent and (ii) normal to the curve
   (a) \( f(x) = 4x^2 - x + 8 \) at the point \((1, 11)\)
   (b) \( y = x^3 + 2x^2 - 5x \) at the point \((-3, 6)\)
   (c) \( F(x) = x^5 - 5x^3 \) at the point where \( x = 1 \)
   (d) \( y = x^2 - 8x + 7 \) at the point \((3, -8)\)
   (e) \( y = x^4 - 2x^3 + 4x + 1 \) at the point where \( x = 1 \).

7. For the curve \( y = x^3 - 27x - 5 \), find values of \( x \) for which \( \frac{dy}{dx} = 0 \).

8. Find the coordinates of the point at which the curve \( y = x^4 + 1 \) has a tangent with a gradient of 3.

9. A function \( f(x) = x^2 + 4x - 12 \) has a tangent with a gradient of -6 at point \( P \) on the curve. Find the coordinates of the point \( P \).

10. The tangent at point \( P \) on the curve \( y = 4x^2 + 1 \) is parallel to the \( x \)-axis. Find the coordinates of \( P \).

11. Find the coordinates of point \( Q \) where the tangent to the curve \( y = 5x^2 - 3x \) is parallel to the line \( 7x - y + 3 = 0 \).
12. Find the coordinates of point S where the tangent to the curve
\[ y = x^2 + 4x - 1 \] is perpendicular to the line \[ 4x + 2y + 7 = 0. \]

13. The curve \( y = 3x^2 - 4 \) has a gradient of 6 at point A.
(a) Find the coordinates of A.
(b) Find the equation of the tangent to the curve at A.

14. A function \( h = 3t^2 - 2t + 5 \) has a tangent at the point where \( t = 2 \). Find the equation of the tangent.

15. A function \( f(x) = 2x^2 - 8x + 3 \) has a tangent parallel to the line \( 4x - 2y + 1 = 0 \) at point \( P \). Find the equation of the tangent at \( P \).

Further Differentiation and Indices

The basic rule for differentiating \( x^n \) works for any rational number \( n \).

Investigation

1. (a) Show that
\[ \frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}. \]
(b) Hence differentiate \( y = \frac{1}{x} \) from first principles.
(c) Differentiate \( y = x^{-1} \) using a short method. Do you get the same answer as 1(b)?

2. (a) Show that \( (\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x}) = h \).
(b) Hence differentiate \( y = \sqrt{x} \) from first principles.
(c) Differentiate \( y = x^{\frac{1}{2}} \) and show that this gives the same answer as 2(b).

We sometimes need to change a function into index form before differentiating.

EXAMPLES

1. Differentiate \( 7^{\frac{1}{x}} \).

Solution

\[
\frac{dy}{dx} = 7^{\frac{1}{x}} \cdot \frac{1}{3} \cdot x^{\frac{1}{3} - 1} \\
= \frac{7}{3} \cdot x^{-\frac{2}{3}} \\
= \frac{7}{3} \cdot \frac{1}{x^{\frac{2}{3}}}
\]
\[ \frac{7}{3} \times \frac{1}{\sqrt{x^2}} = \frac{7}{3\sqrt{x^2}} \]

2. Find the equation of the tangent to the curve \( y = \frac{4}{x^2} \) at the point where \( x = 2 \).

**Solution**

\[
\begin{align*}
y &= \frac{4}{x^2} \\
&= 4x^{-2} \\
dy \\
dx &= -8x^{-3} \\
&= -\frac{8}{x^3}
\end{align*}
\]

When \( x = 2 \)

\[
\begin{align*}
y &= \frac{4}{2^2} \\
&= 1
\end{align*}
\]

Gradient of the tangent at \((2, 1)\):

\[
\begin{align*}
\frac{dy}{dx} &= -\frac{8}{2^3} \\
&= -1
\end{align*}
\]

Equation of the tangent:

\[
\begin{align*}
y - y_1 &= m(x - x_1) \\
y - 1 &= -1(x - 2) \\
&= -x + 2 \\
y &= -x + 3
\end{align*}
\]

or \( x + y - 3 = 0 \)


8.7 **Exercises**

1. Differentiate

(a) \( x^{-3} \)

(b) \( x^{1.4} \)

(c) \( 6x^{0.2} \)

(d) \( x^{\frac{1}{2}} \)

(e) \( 2x^{\frac{1}{2}} - 3x^{-1} \)

(f) \( 3x^{\frac{1}{3}} \)

(g) \( 8x^{\frac{3}{2}} \)

(h) \( -2x^{-\frac{1}{2}} \)

2. Find the derivative function, writing the answer without negative or fractional indices.

(a) \( \frac{1}{x} \)

(b) \( 5\sqrt{x} \)

(c) \( \frac{5}{\sqrt{x}} \)

(d) \( \frac{2}{x^3} \)

(e) \( -\frac{5}{x^3} \)

(f) \( \frac{1}{\sqrt{x}} \)
Note that \( \frac{1}{2x^2} = \frac{1}{2} \times \frac{1}{x^2} \)

Use index laws to simplify first.

(g) \( \frac{1}{2x^5} \)

(h) \( x\sqrt{x} \)

(i) \( \frac{2}{3x} \)

(j) \( \frac{1}{4x^2} + \frac{3}{x^4} \)

3. Find the gradient of the tangent to the curve \( y = \frac{1}{\sqrt{x}} \) at the point where \( x = 27 \).

4. If \( x = \frac{12}{t} \), find \( \frac{dx}{dt} \) when \( t = 2 \).

5. A function is given by \( f(x) = \frac{1}{\sqrt{x}} \). Evaluate \( f'(16) \).

6. Find the gradient of the tangent to the curve \( y = \frac{3}{2x^2} \) at the point \( \left( 1, \frac{1}{2} \right) \).

7. Find \( \frac{dy}{dx} \) if \( y = (x + \sqrt{x})^2 \).

8. A function \( f(x) = \frac{\sqrt{x}}{2} \) has a tangent at \((4, 1)\). Find the gradient of the tangent.

9. Find the equation of the tangent to the curve \( y = \frac{1}{x^2} \) at the point \( \left( 2, \frac{1}{8} \right) \).

10. Find the equation of the tangent to \( f(x) = 6\sqrt{x} \) at the point where \( x = 9 \).

11. (a) Differentiate \( \frac{\sqrt{x}}{x} \).

(b) Hence find the gradient of the tangent to the curve \( y = \frac{\sqrt{x}}{x} \) at the point where \( x = 4 \).

12. Find the equation of the tangent to the curve \( y = \frac{4}{x} \) at the point \( \left( 8, \frac{1}{2} \right) \).

13. If the gradient of the tangent to \( y = \sqrt{x} \) is \( \frac{1}{6} \) at point \( A \), find the coordinates of \( A \).

14. The function \( f(x) = 3\sqrt{x} \) has \( f'(x) = \frac{3}{4} \). Evaluate \( x \).

15. The hyperbola \( y = \frac{2}{x} \) has two tangents with gradient \( -\frac{2}{25} \). Find the coordinates of the points of contact of these tangents.

---

**Composite Function Rule**

A **composite function** is a function composed of two or more other functions. For example, \((3x^2 - 4)^5\) is made up of a function \( u^5 \) where \( u = 3x^2 - 4 \).

To differentiate a composite function, we need to use the result.

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]
Proof

Let $\delta x$, $\delta y$ and $\delta u$ be small changes in $x$, $y$ and $u$ where $\delta x \to 0$, $\delta y \to 0$, $\delta u \to 0$.

Then $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

As $\delta x \to 0$, $\delta u \to 0$

So $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$

Using the definition of the derivative from first principles, this gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$ 

EXAMPLES

Differentiate

1. $(5x + 4)^7$

Solution

Let $u = 5x + 4$

Then $\frac{du}{dx} = 5$

$y = u^7$

$\therefore \frac{dy}{du} = 7u^6$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 7u^6 \times 5$

$= 35(5x + 4)^6$

2. $(3x^2 + 2x - 1)^9$

Solution

Let $u = 3x^2 + 2x - 1$

Then $\frac{du}{dx} = 6x + 2$

$y = u^9$

$\therefore \frac{dy}{du} = 9u^8$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 9u^8(6x + 2)$

$= 9(6x + 2)(3x^2 + 2x - 1)^8$

CONTINUED
3. \( \sqrt{3 - x} \)

Solution

\[
\sqrt{3 - x} = (3 - x)^{\frac{1}{2}}
\]

Let \( u = 3 - x \)

\[
\frac{du}{dx} = -1
\]

\[
y = u^\frac{1}{2}
\]

\[
\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

\[
= \frac{1}{2} u^{-\frac{1}{2}} (-1)
\]

\[
= -\frac{1}{2} (3 - x)^{-\frac{1}{2}}
\]

\[
= -\frac{1}{2 \sqrt{3 - x}}
\]

The derivative of a composite function is the product of two derivatives. One is the derivative of the function inside the brackets. The other is the derivative of the whole function.

\[
\frac{d}{dx} (f(x))^n = f'(x) n [f(x)]^{n-1}
\]

**Proof**

Let \( u = f(x) \)

Then \( \frac{du}{dx} = f'(x) \)

\[
y = u^n
\]

\[
\frac{dy}{du} = nu^{n-1}
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

\[
= nu^{n-1} \times f'(x)
\]

\[
= f'(x) n [f(x)]^{n-1}
\]
EXAMPLES

Differentiate

1. \((8x^3 - 1)^5\)

Solution

\[
\frac{dy}{dx} = f'(x) \cdot n [f(x)]^{n-1}
\]
\[
= 24x^2 \cdot 5 (8x^3 - 1)^4
\]
\[
= 120x^2(8x^3 - 1)^4
\]

2. \((3x + 8)^{11}\)

Solution

\[
y' = f'(x) \cdot n [f(x)]^{n-1}
\]
\[
= 3 \times 11 (3x + 8)^{10}
\]
\[
= 33(3x + 8)^{10}
\]

3. \(\frac{1}{(6x + 1)^2}\)

Solution

\[
\frac{1}{(6x + 1)^2} = (6x + 1)^{-2}
\]
\[
y' = f'(x) \cdot n [f(x)]^{n-1}
\]
\[
= 6 \times -2(6x + 1)^{-3}
\]
\[
= -12(6x + 1)^{-3}
\]
\[
= -\frac{12}{(6x + 1)^3}
\]

8.8 Exercises

1. Differentiate
   (a) \((x + 3)^4\)
   (b) \((2x - 1)^5\)
   (c) \((5x^2 - 4)^7\)
   (d) \((8x + 3)^6\)
   (e) \((1 - x)^5\)
   (f) \(3(5x + 9)^9\)
   (g) \(2(x - 4)^2\)
   (h) \((2x^3 + 3x)^4\)
   (i) \((x^2 + 5x - 1)^8\)
   (j) \((x^6 - 2x^2 + 3)^6\)
   (k) \((3x - 1)^{\frac{1}{2}}\)
(l) \((4 - x)^{-2}\)
(m) \((x^2 - 9)^{-3}\)
(n) \((5x + 4)^{\frac{1}{3}}\)
(o) \((x^3 - 7x^2 + x)^{\frac{3}{2}}\)
(p) \(\sqrt{3x + 4}\)
(q) \(\frac{1}{5x - 2}\)
(r) \(\frac{1}{(x^2 + 1)^4}\)
(s) \(\frac{1}{\sqrt[3]{7 - 3x}}^2\)
(t) \(\frac{5}{\sqrt{4 + x}}\)
(u) \(\frac{1}{2\sqrt{3x} - 1}\)
(v) \(\frac{3}{4(2x + 7)^9}\)
(w) \(\frac{1}{x^4 - 3x^3 + 3x}\)
(x) \(\frac{1}{\sqrt[4]{4x + 1}}^4\)
(y) \(\frac{1}{\sqrt[5]{7 - x}}^5\)

2. Find the gradient of the tangent to the curve \(y = (3x - 2)^{\frac{1}{2}}\) at the point \((1, 1)\).

3. If \(f(x) = 2(x^2 - 3)^{\frac{5}{3}}\), evaluate \(f'(2)\).

4. The curve \(y = \sqrt{x - 3}\) has a tangent with gradient \(\frac{1}{2}\) at point \(N\). Find the coordinates of \(N\).

5. For what values of \(x\) does the function \(f(x) = \frac{1}{4x - 1}\) have \(f'(x) = -\frac{4}{49}\)?

6. Find the equation of the tangent to \(y = (2x + 1)^4\) at the point where \(x = -1\).

Product Rule

Differentiating the product of two functions \(y = uv\) gives the result

\[
\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
\]

Proof

Given that \(\delta y, \delta u\) and \(\delta v\) are small changes in \(y, u\) and \(v\).

\[y + \delta y = (u + \delta u)(v + \delta v)\]
\[= uv + u\delta v + v\delta u + \delta u\delta v\]

\[\therefore \delta y = u\delta v + v\delta u + \delta u\delta v \text{ (since } y = uv\)]

\[
\frac{\delta y}{\delta x} = u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \delta u \frac{\delta v}{\delta x}
\]
As $\delta x \to 0, \delta u \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[ u \frac{\delta y}{\delta x} + v \frac{\delta u}{\delta x} + \delta u \frac{\delta v}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} \left[ u \frac{\delta y}{\delta x} \right] + \lim_{\delta x \to 0} \left[ v \frac{\delta u}{\delta x} \right] + \lim_{\delta x \to 0} \left[ \delta u \frac{\delta v}{\delta x} \right]$$

$$\frac{dy}{dx} = u \frac{dy}{dx} + v \frac{du}{dx}$$

It is easier to remember this rule as $y' = uv' + vu'$. We can also write this the other way around which helps when learning the quotient rule in the next section.

If $y = uv$, $y' = u'v + v'u$

### EXAMPLES

**Differentiate**

1. $(3x + 1)(x - 5)$

**Solution**

You could expand the brackets and then differentiate:

$(3x + 1)(x - 5) = 3x^2 - 15x + x - 5$

$= 3x^2 - 14x - 5$

$$\frac{dy}{dx} = 6x - 14$$

Using the product rule:

$y = uv$ where $u = 3x + 1$ and $v = x - 5$

$u' = 3$ \quad $v' = 1$

$y' = u'v + v'u$

$= 3(x - 5) + 1(3x + 1)$

$= 3x - 15 + 3x + 1$

$= 6x - 14$

2. $2x^3(5x + 3)^3$

**Solution**

$y = uv$ where $u = 2x^3$ and $v = (5x + 3)^3$

$u' = 10x^2$ \quad $v' = 5.3(5x + 3)^2$
\[ y' = u'v + v'u \]
\[ = 10x^4(5x + 3)^3 + 5.3(5x + 3)^2 \cdot 2x^3 \]
\[ = 10x^4(5x + 3)^3 + 30x^3(5x + 3)^2 \]
\[ = 10x^4(5x + 3)^2[(5x + 3) + 3x] \]
\[ = 10x^4(5x + 3)^2(8x + 3) \]

3. \( (3x - 4)\sqrt{5 - 2x} \)

**Solution**

Remember \( \sqrt{5 - 2x} = (5 - 2x)^{\frac{1}{2}} \)

\[ y = uv \text{ where } u = 3x - 4 \text{ and } v = (5 - 2x)^{\frac{1}{2}} \]
\[ u' = 3 \quad v' = -2 \cdot \frac{1}{2}(5 - 2x)^{-\frac{1}{2}} \]

\[ y' = u'v + v'u \]
\[ = 3(5 - 2x)^{\frac{1}{2}} + -2 \cdot \frac{1}{2}(5 - 2x)^{-\frac{1}{2}}(3x - 4) \]
\[ = 3\sqrt{5 - 2x} - (3x - 4)(5 - 2x)^{-\frac{1}{2}} \]
\[ = 3\sqrt{5 - 2x} - \frac{3x - 4}{\sqrt{5 - 2x}} \]
\[ = 3\sqrt{5 - 2x} \cdot \sqrt{5 - 2x} - (3x - 4) \]
\[ = \frac{\sqrt{5 - 2x}}{3(5 - 2x) - (3x - 4)} \]
\[ = \frac{\sqrt{5 - 2x}}{15 - 6x - 3x + 4} \]
\[ = \frac{19 - 9x}{\sqrt{5 - 2x}} \]

8.9 **Exercises**

1. **Differentiate**

(a) \( x^3(2x + 3) \)
(b) \( (3x - 2)(2x + 1) \)
(c) \( 3x(5x + 7) \)
(d) \( 4x^4(3x^2 - 1) \)
(e) \( 2x(3x^4 - x) \)
(f) \( x^2(x + 1)^3 \)

(g) \( 4x(3x - 2)^3 \)
(h) \( 3x^4(4 - x)^3 \)
(i) \( (x + 1)(2x + 5)^4 \)
(j) \( x^3 + 5x^2 - 3)(x^2 + 1)^5 \)
(k) \( x\sqrt{2 - x} \)
(l) \( \frac{5x + 3}{2x - 1} \)
2. Find the gradient of the tangent to the curve \( y = 2x(3x - 2)^4 \) at the point \((1, 2)\).

3. If \( f(x) = (2x + 3)(3x - 1)^5 \), evaluate \( f'(1) \).

4. Find the exact gradient of the tangent to the curve \( y = x\sqrt{2x + 5} \) at the point where \( x = 1 \).

5. Find the gradient of the tangent where \( t = 3 \), given \( x = (2t - 5)(t + 1)^3 \).

6. Find the equation of the tangent to the curve \( y = x^2(2x - 1)^4 \) at the point \((1, 1)\).

7. Find the equation of the tangent to \( h = (t + 1)^2(t - 1)^7 \) at the point \((2, 9)\).

8. Find exact values of \( x \) for which the gradient of the tangent to the curve \( y = 2x(x + 3)^2 \) is 14.

9. Given \( f(x) = (4x - 1)(3x + 2)^2 \), find the equation of the tangent at the point where \( x = -1 \).

Quotient Rule

Differentiating the quotient of two functions \( y = \frac{u}{v} \) gives the result.

\[
\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

Proof

\[ y = \frac{u}{v} \]

Given that \( \delta y, \delta u \) and \( \delta v \) are small changes in \( y, u \) and \( v \).

\[ y + \delta y = \frac{u + \delta u}{v + \delta v} \]

\[ \therefore \delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} \quad \text{(since } y = \frac{u}{v}\text{)} \]

\[ \delta y = \frac{v(u + \delta u)}{v(v + \delta v)} - \frac{u(v + \delta v)}{v(v + \delta v)} \]

\[ = \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)} \]

\[ = \frac{v\delta u - u\delta v}{v(v + \delta v)} \]

\[ \frac{\delta y}{\delta x} = \frac{v\delta u - u\delta v}{v(v + \delta v)} \]

As \( \delta x \to 0, \delta v \to 0 \)
\[
\lim_{\Delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\Delta x \to 0} \left[ \frac{\delta u}{\delta x} - \frac{\delta v}{\delta x} \right] \frac{v + \delta v}{v} \\\text{or} \\\frac{v du}{dx} - \frac{u dv}{dx} \over v^2
\]

It is easier to remember this rule as \( y' = \frac{u'v - v'u}{v^2} \).

If \( y = \frac{u}{v} \), \( y' = \frac{u'v - v'u}{v^2} \)

**EXAMPLES**

Differentiate

1. \( \frac{3x - 5}{5x + 2} \)

**Solution**

\( y = \frac{u}{v} \) where \( u = 3x - 5 \) and \( v = 5x + 2 \)

\( u' = 3 \quad v' = 5 \)

\( y' = \frac{u'v - v'u}{v^2} \)

\( = \frac{3(5x + 2) - 5(3x - 5)}{(5x + 2)^2} \)

\( = \frac{15x + 6 - 15x + 25}{(5x + 2)^2} \)

\( = \frac{31}{(5x + 2)^2} \)

2. \( \frac{4x^3 - 5x + 2}{x^3 - 1} \)

**Solution**

\( y = \frac{u}{v} \) where \( u = 4x^3 - 5x + 2 \) and \( v = x^3 - 1 \)

\( u' = 12x^2 - 5 \quad v' = 3x^2 \)

\( y' = \frac{u'v - v'u}{v^2} \)

\( = \frac{(12x^2 - 5)(x^3 - 1) - 3x^2(4x^3 - 5x + 2)}{(x^3 - 1)^2} \)

\( = \frac{12x^5 - 12x^2 - 5x^4 + 5 - 12x^3 + 15x^3 - 6x^2}{(x^3 - 1)^2} \)

\( = \frac{10x^3 - 18x^2 + 5}{(x^3 - 1)^2} \)
8.10 Exercises

1. Differentiate

(a) \( \frac{1}{2x - 1} \)
(b) \( \frac{3x}{x + 5} \)
(c) \( \frac{x^3}{x^2 - 4} \)
(d) \( \frac{x - 3}{5x + 1} \)
(e) \( \frac{x - 7}{x^2} \)
(f) \( \frac{5x + 4}{x + 3} \)
(g) \( \frac{x}{2x^2 - x} \)
(h) \( \frac{x + 4}{x - 2} \)
(i) \( \frac{2x + 7}{4x - 3} \)
(j) \( \frac{x + 5}{3x + 1} \)
(k) \( \frac{x + 1}{3x^2 - 7} \)
(l) \( \frac{2x^2}{2x - 3} \)
(m) \( \frac{x^2 + 4}{x^2 - 5} \)
(n) \( \frac{x^3}{x + 4} \)
(o) \( \frac{x^3 + 2x - 1}{x + 3} \)
(p) \( \frac{x^2 - 2x - 1}{3x + 4} \)
(q) \( \frac{x^3 + x}{x^2 - x - 1} \)
(r) \( \frac{2x}{(x + 5)^3} \)
(s) \( \frac{(2x - 9)^3}{5x + 1} \)
(t) \( \frac{x - 1}{(7x + 2)^4} \)
(u) \( \frac{(3x + 4)^5}{(2x - 5)^3} \)
(v) \( \frac{3x + 1}{\sqrt{x + 1}} \)
(w) \( \frac{x - 1}{2x - 3} \)
(x) \( \frac{x^2 + 1}{(x - 9)^2} \)

2. Find the gradient of the tangent to the curve \( y = \frac{2x}{3x + 1} \) at the point \( \left( 1, \frac{1}{2} \right) \).

3. If \( f(x) = \frac{4x + 5}{2x - 1} \) evaluate \( f'(2) \).

4. Find any values of \( x \) for which the gradient of the tangent to the curve \( y = \frac{4x - 1}{2x - 1} \) is equal to \(-2\).

5. Given \( f(x) = \frac{2x}{x + 3} \) find \( x \) if \( f'(x) = \frac{1}{6} \).

6. Find the equation of the tangent to the curve \( y = \frac{x}{x + 2} \) at the point \( \left( 4, \frac{2}{3} \right) \).

7. Find the equation of the tangent to the curve \( y = \frac{x^2 - 1}{x + 3} \) at the point where \( x = 2 \).

Angle Between 2 Curves

To measure the angle between two curves, measure the angle between the tangents to the curves at that point.
EXAMPLE

Find the acute angle formed at the intersection of the curves $y = x^2$ and $y = (x - 2)^2$.

Solution

The curves intersect at the point $(1,1)$.

For $y = x^2$

$$\frac{dy}{dx} = 2x$$

At $(1,1)$, $\frac{dy}{dx} = 2(1)$

$\therefore m_1 = 2$

For $y = (x - 2)^2$

$$\frac{dy}{dx} = 2(x - 2)$$

At $(1,1)$, $\frac{dy}{dx} = 2(1 - 2)$

$\therefore m_2 = -2$

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1m_2} \right|$$

$$= \left| \frac{2 - (-2)}{1 + 2(-2)} \right|$$

$$= \frac{4}{3}$$

$\theta = 53^\circ 08'$
1. (a) Sketch the curves $y = x^2 - 4$ and $y = x^2 - 8x + 12$ on the same set of axes.
(b) Show that the curves intersect at the point $Q(2, 0)$.
(c) Find the gradient of the tangent of each curve at point $Q$.
(d) Find the acute angle at which the curves intersect at $Q$.

2. (a) Sketch the curve $y = x^2$ and the line $y = 6x - 9$ on the same set of axes.
(b) Find the point $P$, their point of intersection.
(c) Find the gradient of the curve $y = x^2$ at $P$.
(d) Find the acute angle between the curve and the line at $P$.

3. Find the acute angle between the curves $y = x^2$ and $y = x^3$ at point $(1, 1)$.

4. Find the acute angle between the curves $y = x^4$ and $y = x^2 - 2x + 2$ at their point of intersection.

5. What is the obtuse angle between the curves $f(x) = x^2 - 4x$ and $g(x) = x^2 - 12$ at the point where they meet?

6. The curves $y = 2x^2 - 4x$ and $y = x^2 - x + 4$ intersect at two points $X$ and $Y$.
(a) Find the coordinates of $X$ and $Y$.
(b) Find the gradient of the tangent to each curve at $X$ and $Y$.
(c) Find the acute angle between the curves at $X$ and $Y$.

7. Find the acute angle between the curve $f(x) = x^2 - 1$ and the line $g(x) = 3x - 1$ at their 2 points of intersection.

8. (a) Find the points of intersection between $y = x^3$ and $y = x^2 + 2x$.
(b) Find the acute angle between the curves at these points.

9. Show that the acute angle between the curves $y = x^2$ and $y = 4x - x^2$ is the same at both the points of intersection.

10. Find the obtuse angles between the curves $y = x^3 + 2x$ and $y = 5x - 2x^2$ at their points of intersection.
1. Sketch the derivative function of each graph
   (a)
   ![Graph](image1)
   (b)
   ![Graph](image2)

2. Differentiate \( y = 5x^2 - 3x + 2 \) from first principles.

3. Differentiate
   (a) \( 7x^6 - 3x^3 + x^2 - 8x - 4 \)
   (b) \( \frac{3x - 4}{2x + 1} \)
   (c) \((x^2 + 4x - 2)^9\)
   (d) \(5x(2x - 1)^4\)
   (e) \(x^2 \sqrt{x}\)
   (f) \(\frac{5}{x^2}\)

4. Find \( \frac{dv}{dt} \) if \( v = 2t^2 - 3t - 4 \).

5. Given \( f(x) = (4x - 3)^2 \), find the value of
   (a) \( f(1) \)
   (b) \( f^1(1) \).

6. Find the gradient of the tangent to the curve \( y = x^3 - 3x^2 + x - 5 \) at the point \((-1, -10)\).

7. If \( h = 60t - 3t^2 \), find \( \frac{dh}{dt} \) when \( t = 3 \).

8. Find all \( x \)-values that are not differentiable on the following curves.
   (a)
   ![Graph](image3)
   (b)
   ![Graph](image4)
   (c)
   ![Graph](image5)

9. Differentiate
   (a) \( f'(x) = 2(4x + 9)^4 \)
   (b) \( y = \frac{5}{x - 3} \)
   (c) \( y = x(3x - 1)^2 \)
   (d) \( y = \frac{4}{x} \)
   (e) \( f(x) = \sqrt[3]{x} \)
10. Sketch the derivative function of the following curve.

![Graph of the function](image)

11. Find the equation of the tangent to the curve \( y = x^2 + 5x - 3 \) at the point \((2, 11)\).

12. Find the point on the curve \( y = x^2 - x + 1 \) at which the tangent has a gradient of 3.

13. Find \( \frac{dS}{dr} \) if \( S = 4\pi r^2 \).

14. At which points on the curve \( y = 2x^3 - 9x^2 - 60x + 3 \) are the tangents horizontal?

15. Find the equation of the tangent to the curve \( y = x^2 + 2x - 5 \) that is parallel to the line \( y = 4x - 1 \).

16. Find the gradient of the tangent to the curve \( y = (3x - 1)^3(2x - 1)^2 \) at the point where \( x = 2 \).

17. Find \( f'(4) \) when \( f(x) = (x - 3)^9 \).

18. Find the equation of the tangent to the curve \( y = \frac{1}{3x} \) at the point where \( x = \frac{1}{6} \).

19. Differentiate \( s = ut + \frac{1}{2}at^2 \) with respect to \( t \) and find the value of \( t \) for which \( \frac{ds}{dt} = 5 \), \( u = 7 \) and \( a = -10 \).

20. Find the \( x \)-intercept of the tangent to the curve \( y = \frac{4x - 3}{2x + 1} \) at the point where \( x = 1 \).

21. Find the acute angle between the curve \( y = x^2 \) and the line \( y = 2x + 3 \) at each point of intersection.

22. Find the obtuse angle between the curve \( y = x^2 \) and the line \( y = 6x - 8 \) at each point of intersection.

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### Challenge Exercise 8

1. If \( f(x) = 3x^2(1 - 2x)^3 \), find the value of \( f(1) \) and \( f'(1) \).

2. If \( A = \frac{5h + 3}{7h - 1} \), find \( \frac{dA}{dh} \) when \( h = 1 \).

3. Given \( x = 2t^4 + 100t^3 \), find \( \frac{dx}{dt} \) and find values of \( t \) when \( \frac{dx}{dt} = 0 \).

4. Find the equations of the tangents to the curve \( y = x(x - 1)(x + 2) \) at the points where the curve cuts the \( x \)-axis.

5. Find the points on the curve \( y = x^3 - 6 \) where the tangents are parallel to the line \( y = 12x - 1 \). Hence find the equations of the normals to the curve at those points.
6. Find $f''(2)$ if $f(x) = \sqrt[3]{3x - 2}$.

7. Differentiate $(5x + 1)^5(x - 9)^5$.

8. Find the derivative of $y = \frac{2x + 1}{(4x - 9)^4}$.

9. If $f(x) = 2x^3 + 3x^2 + 4$, for what exact values of $x$ is $f'(x) = 7$?

10. Find the equation of the normal to the curve $y = 3\sqrt{x + 1}$ at the point where $x = 8$.

11. The tangent to the curve $y = ax^3 + 2$ at the point where $x = 3$ is inclined at 135° to the $x$-axis. Find the value of $a$.

12. The normal to the curve $y = x^2 + 1$ at the point where $x = 2$, cuts the curve again at point $P$. Find the coordinates of $P$.

13. Find the exact values of the $x$-coordinates of the points on the curve $y = (3x^2 - 2x - 4)^3$ where the tangent is horizontal.

14. Find the gradient of the normal to the curve $y = 2x\sqrt{5 - x}$ at the point $(4, 8)$.

15. Find the equation of the tangent to the curve $y = x^3 - x^2 + 2x + 6$ at point $P(1, 8)$. Find the coordinates of point $Q$ where this tangent meets the $y$-axis and calculate the exact length of $PQ$.

16. (a) Show that the curves $y = (3x - 2)^5$ and $y = \frac{5x - 3}{x + 1}$ intersect at $(1, 1)$

(b) Find the acute angle between the curves at this point.

17. The equation of the tangent to the curve $y = x^4 - nx^3 + 3x - 2$ at the point where $x = -2$ is given by $3x - y - 2 = 0$. Evaluate $n$.

18. The function $f(x) = \sqrt[3]{3x + 1}$ has a tangent that makes an angle of 30° with the $x$-axis. Find the coordinates of the point of contact for this tangent and find its equation in exact form.

19. Find all $x$ values of the function $f(x) = (x^2 - 3)(2x - 1)^9$ for which $f'(x) = 0$.

20. (a) Find any points at which the graph below is not differentiable.

(b) Sketch the derivative function for the graph.

21. Find the point of intersection between the tangents to the curve $y = x^4 - 2x^2 - 5x + 3$ at the points where $x = 2$ and $x = -1$.

22. Find the equation of the tangent to the parabola $y = \frac{x^2 - 3}{2}$ at the point where the tangent is perpendicular to the line $3x + y - 3 = 0$.

23. Differentiate $\frac{\sqrt{3x - 2}}{2x^3}$.

24. (a) Find the equations of the tangents to the parabola $y = 2x^2$ at the points where the line $6x - 8y + 1 = 0$ intersects with the parabola.

(b) Show that the tangents are perpendicular.
25. Find any \( x \) values of the function
\[ f(x) = \frac{2}{x^3 - 8x^2 + 12x} \]
where it is not differentiable.

26. The equation of the tangent to the curve
\[ y = x^3 + 7x^2 - 6x - 9 \]
is \( y = ax + b \) at the point where \( x = -4 \). Evaluate \( a \) and \( b \).

27. Find the exact gradient with rational denominator of the tangent to the curve
\[ y = \sqrt{x^2 - 3} \]
at the point where \( x = 5 \).

28. The tangent to the curve \( y = \frac{p}{x} \) has a gradient of \( -\frac{1}{6} \) at the point where \( x = 3 \). Evaluate \( p \).

29. Find \( \frac{dV}{dr} \) when \( r = \frac{2\pi}{3} \) and \( h = 6 \) given
\[ V = \frac{1}{3}\pi r^3 h. \]

30. Evaluate \( k \) if the function
\[ f(x) = 2x^3 - kx^2 + 1 \]
has \( f'(2) = 8 \).

31. Find the equation of the chord joining the points of contact of the tangents to the curve \( y = x^2 - x - 4 \) with gradients 3 and -1.

32. Find the equation of the straight line passing through \((4, 3)\) and parallel to the tangent to the curve \( y = x^4 \) at the point \((1, 1)\).

33. Find \( f'(7) \) as a fraction, given
\[ f(x) = \frac{1}{\sqrt{x + 1}}. \]

34. For the function
\[ f(x) = ax^2 + bx + c, \]
f(2) = 4, \( f'(1) = 0 \) and \( f'(x) = 8 \) when \( x = -3 \). Evaluate \( a, b \) and \( c \).

35. Find the equation of the tangent to the curve \( S = 2\pi r^2 + 2\pi rh \) at the point where \( r = 2 \) (\( h \) is a constant).

36. Differentiate
(a) \( 2x^3 - x(3x - 5)^4 \)
(b) \( \sqrt{2x + 1} \)
\( (x - 3)^{3} \)

37. The tangents to the curve
\[ y = x^3 - 2x^2 + 3 \]
at points \( A \) and \( B \) are perpendicular to the tangent at \( (2, 3) \). Find the exact values of \( x \) at \( A \) and \( B \).

38. (a) Find the equation of the normal to the curve \( y = x^3 + x - 1 \) at the point \( P \) where \( x = 3 \).
(b) Find the coordinates of \( Q \), the point where the normal intersects the parabola again.